

# Cubic identities for theta series in three variables

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## 1 Introduction

In [1] (see also [2]) Borwein and Borwein proved the identity

$$a(q)^3 = b(q)^3 + c(q)^3 \quad (1)$$

where

$$a(q) = \sum_{m,n \in \mathbf{Z}} q^{m^2+mn+n^2},$$

$$b(q) = \sum_{m,n \in \mathbf{Z}} \omega^{m-n} q^{m^2+mn+n^2}$$

and

$$c(q) = \sum_{m,n \in \mathbf{Z}} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}$$

where  $\omega = \exp(2\pi i/3)$ . We call these functions theta series for convenience. Subsequently Hirschhorn, Garvan and J. Borwein [3] proved the corresponding identity for two-variable analogues of these theta series. Solé [4] (see also [5]) gave a new proof of (1) using a lattice having the structure of a  $\mathbf{Z}[\omega]$ -module. Here we introduce three-variable analogues of the theta series  $a(q)$ ,  $b(q)$  and  $c(q)$ , and adapt Solé's method to prove corresponding identities for them.

## 2 Theta series

We introduce our three-variable theta series as sums over elements of the Eisenstein field  $\mathbf{Q}(\sqrt{-3})$ .

Let  $K = \mathbf{Q}(\sqrt{-3})$  and let  $\mathcal{O} = \mathbf{Z}[\omega]$  be its ring of integers, where  $\omega = \frac{1}{2}(-1 + \sqrt{-3}) = \exp(2\pi i/3)$ . Write  $\lambda = \omega - \omega^2 = \sqrt{-3}$ . For  $\alpha \in K$  define  $T(\alpha) = \alpha + \bar{\alpha}$ , the trace of  $\alpha$ . The element  $\lambda$  generates a prime ideal of  $\mathcal{O}$  of norm 3; the inclusion  $\mathbf{Z} \rightarrow \mathcal{O}$  induces an isomorphism  $\mathbf{Z}/3\mathbf{Z} \cong \mathcal{O}/\lambda\mathcal{O}$ . Hence we can unambiguously define, for  $\alpha \in \mathcal{O}$ ,  $\chi(\alpha) = \omega^a$  where  $a \in \mathbf{Z}$  and  $\alpha \equiv a \pmod{\lambda\mathcal{O}}$ .

We now define our theta series. We start with

$$a(q, z, w) = \sum_{\alpha \in \mathcal{O}} q^{|\alpha|^2} z^{T(\alpha)} w^{T(\alpha/\lambda)}.$$

Next for any integer  $k$  define

$$b_k(q, z, w) = \sum_{\alpha \in \mathcal{O}} \chi(\alpha)^k q^{|\alpha|^2} z^{T(\alpha)} w^{T(\alpha/\lambda)}.$$

It is apparent that  $b_k(q, z, w)$  depends only on the congruence class of  $k$  modulo 3 and that  $b_0(q, z, w) = a(q, z, w)$ . We also define

$$c_k(q, z, w) = \sum_{\alpha \in \mathcal{O} + k/\lambda} q^{|\alpha|^2} z^{T(\alpha)} w^{T(\alpha/\lambda)}.$$

Again  $c_k(q, z, w)$  depends only on the congruence class of  $k$  modulo 3 and  $c_0(q, z, w) = a(q, z, w)$ .

We observe some symmetry properties of these functions.

**Lemma 1** *We have*

$$a(q, z, w) = a(q, z, w^{-1}) = a(q, z^{-1}, w^{-1}) = a(q, z^{-1}, w), \quad (2)$$

$$b_k(q, z, w) = b_k(q, z, w^{-1}) = b_{-k}(q, z^{-1}, w^{-1}) = b_{-k}(q, z^{-1}, w) \quad (3)$$

and

$$c_k(q, z, w) = c_{-k}(q, z, w^{-1}) = c_{-k}(q, z^{-1}, w^{-1}) = c_k(q, z^{-1}, w). \quad (4)$$

**Proof** We replace  $\alpha$  in the definition of each series in turn by  $\bar{\alpha}$ ,  $-\alpha$  and  $-\bar{\alpha}$ . It helps to note that  $T(\bar{\alpha}) = T(\alpha)$ ,  $T(\bar{\alpha}/\lambda) = -T(\alpha/\lambda)$ ,  $\chi(\bar{\alpha}) = \chi(\alpha)$ ,  $\chi(-\alpha) = \chi(\alpha)^{-1}$  and  $\overline{k/\lambda} = -k/\lambda$ . Of course (2) is a special case of both (3) and (4).  $\square$

From (3) and (4) we see that  $b_1(q, 1, 1) = b_{-1}(q, 1, 1)$  and  $c_1(q, 1, 1) = c_{-1}(q, 1, 1)$ . We write

$$a(q) = a(q, 1, 1), \quad b(q) = b_1(q, 1, 1) \quad \text{and} \quad c(q) = c_1(q, 1, 1).$$

We shall soon see that this agrees with our previous definition.

We show that these functions specialize to the two-variable functions introduced in [3]. First of all, each element  $\alpha \in \mathcal{O}$  can be uniquely written as  $\alpha = n\omega - m\omega^2$ . Then  $T(\alpha) = m - n$  and

$$|\alpha|^2 = (n\omega - m\omega^2)(n\omega^2 - m\omega) = m^2 + mn + n$$

and so

$$a(q, z, 1) = \sum_{m, n \in \mathbf{Z}} q^{m^2 + mn + n^2} z^{m-n}$$

which is denoted as  $a(q, z)$  in [3]. In particular

$$a(q, 1, 1) = \sum_{m, n \in \mathbf{Z}} q^{m^2 + mn + n^2}$$

in agreement with the original definition. Also  $|\omega\alpha|^2 = |\alpha|^2$  and  $T(-\omega\alpha) = T((m - n\omega^2)/\lambda) = n$ . Hence

$$a(q, 1, z) = \sum_{\alpha \in \mathcal{O}} q^{|\omega\alpha|^2} z^{T(-\omega\alpha/\lambda)} = \sum_{m, n \in \mathbf{Z}} q^{m^2 + mn + n^2} z^n$$

which is denoted as  $a'(q, z)$  in [3]. Now  $\chi(-\omega\alpha) = \omega^{m-n}$  and similarly

$$b_1(q, 1, z) = \sum_{m, n \in \mathbf{Z}} \omega^{m-n} q^{m^2 + mn + n^2} z^n$$

which is denoted as  $b(q, z)$  in [3]. In particular

$$b_1(q, 1, 1) = \sum_{m, n \in \mathbf{Z}} \omega^{m-n} q^{m^2 + mn + n^2}.$$

Note that  $b_1(q, 1, z) = b_{-1}(q, 1, z)$  by (3). Finally  $\frac{1}{3}(\omega - \omega^2) = -1/\lambda$  and so

$$c_{-1}(q, z, 1) = \sum_{m, n \in \mathbf{Z}} q^{(m+1/3)^2 + (m+1/3)(n+1/3) + (n+1/3)^2} z^{m-n}$$

and this is denoted by  $q^{1/3}c(q, z)$  in [3]. Again note that  $c_{-1}(q, z, 1) = c_1(q, z, 1)$  by (3). In particular

$$c_1(q, 1, 1) = c_{-1}(q, 1, 1) = \sum_{m, n \in \mathbf{Z}} q^{(m+1/3)^2 + (m+1/3)(n+1/3) + (n+1/3)^2}.$$

### 3 Identities

Our main result is a generalization of (1.25) in [3].

**Theorem 1** *For each integer  $k$ ,*

$$\begin{aligned} 3c_k(q, z, w)^3 &= a(q, w, z^{-3})a(q)^2 \\ &\quad + \omega^k b_1(q, w, z^{-3})b(q)^2 + \omega^{-k} b_{-1}(q, w, z^{-3})b(q)^2 \\ &\quad + c_1(q, w, z^{-3})c(q)^2 + c_{-1}(q, w, z^{-3})c(q)^2. \end{aligned} \quad (5)$$

*In particular*

$$\begin{aligned} 3a(q, z, w)^3 &= a(q, w, z^{-3})a(q)^2 + b_1(q, w, z^{-3})b(q)^2 + b_{-1}(q, w, z^{-3})b(q)^2 \\ &\quad + c_1(q, w, z^{-3})c(q)^2 + c_{-1}(q, w, z^{-3})c(q)^2. \end{aligned} \quad (6)$$

**Proof** Cubing the definition of  $c_k(q, z, w)$  gives

$$c_k(q, z, w)^3 = \sum_{\alpha_0, \alpha_1, \alpha_2 \in \mathcal{O} + k/\lambda} q^{|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2} z^{T(\alpha_0 + \alpha_1 + \alpha_2)} w^{T((\alpha_0 + \alpha_1 + \alpha_2)/\lambda)}. \quad (7)$$

This is a sum over triples  $\alpha = (\alpha_0, \alpha_1, \alpha_2)$  where  $\alpha$  runs through a certain subset of

$$\Lambda = \mathcal{O}^3 + \mathbf{Z}(1/\lambda, 1/\lambda, 1/\lambda).$$

We partition the group  $\Lambda$  into various cosets. If  $\alpha \in \Lambda$  then  $\alpha_0 + \alpha_1 + \alpha_2 \in \mathcal{O}$ . For integers  $j$  and  $k$  let

$$\Lambda_{j,k} = \{\alpha \in \mathcal{O}^3 + k(1/\lambda, 1/\lambda, 1/\lambda) : \alpha_0 + \alpha_1 + \alpha_2 \equiv j \pmod{\lambda}\}.$$

Then  $\Lambda_{j,k}$  depends only on the integers  $j$  and  $k$  modulo 3. They are the nine cosets of the subgroup  $\Lambda_{0,0}$  of  $\Lambda$ . Define, for  $\alpha \in K^3$ ,

$$|\alpha|^2 = |\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2$$

and

$$\Phi(\alpha) = q^{|\alpha|^2} z^{T(\alpha_0 + \alpha_1 + \alpha_2)} w^{T((\alpha_0 + \alpha_1 + \alpha_2)/\lambda)}.$$

Then

$$c_k(q, z, w)^3 = \sum_{\alpha \in \Lambda_{0,k} \cup \Lambda_{1,k} \cup \Lambda_{-1,k}} \Phi(\alpha). \quad (8)$$

We now consider the matrix

$$M = \frac{1}{\lambda} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

It is straightforward to check that  $\Lambda_{j,k}M = \Lambda_{-k,j}$ . Also  $M$  is a unitary matrix so that if  $\beta = \alpha M$  then  $|\beta|^2 = |\alpha|^2$ . Thus

$$\Phi(\alpha) = q^{|\beta|^2} z^{T(\lambda\beta_0)} w^{T(\beta_0)} = q^{|\beta|^2} z^{-3T(\beta_0/\lambda)} w^{T(\beta_0)}.$$

From (8) we get

$$c_k(q, z, w)^3 = \sum_{\beta \in \Lambda_{-k,0} \cup \Lambda_{-k,1} \cup \Lambda_{-k,-1}} q^{|\beta|^2} z^{-3T(\beta_0/\lambda)} w^{T(\beta_0)}. \quad (9)$$

We split this sum into sums over each of the three cosets  $\Lambda_{-k,j}$ .

Consider  $\Lambda_{-k,0}$ . This can be written as

$$\Lambda_{-k,0} = \{\beta \in \mathcal{O}^3 : \chi(\beta_0)\chi(\beta_1)\chi(\beta_2) = \omega^{-k}\}.$$

Hence

$$\begin{aligned} & 3 \sum_{\beta \in \Lambda_{-k,0}} q^{|\beta|^2} z^{T(-3\beta_0/\lambda)} w^{T(\beta_0)} \\ = & \sum_{\beta \in \mathcal{O}^3} q^{|\beta|^2} z^{T(-3\beta_0/\lambda)} w^{T(\beta_0)} \\ & + \omega^k \sum_{\beta \in \mathcal{O}^3} \chi(\beta_0)\chi(\beta_1)\chi(\beta_2) q^{|\beta|^2} z^{T(-3\beta_0/\lambda)} w^{T(\beta_0)} \\ & + \omega^{-k} \sum_{\beta \in \mathcal{O}^3} \chi(\beta_0)^{-1}\chi(\beta_1)^{-1}\chi(\beta_2)^{-1} q^{|\beta|^2} z^{T(-3\beta_0/\lambda)} w^{T(\beta_0)} \\ = & a(q, w, z^{-3})a(q)^2 + \omega^k b_1(q, w, z^{-3})b(q)^2 + \omega^{-k} b_{-1}(q, w, z^{-3})b(q)^2 \end{aligned} \quad (10)$$

To aid with the remaining cosets consider the matrix

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

Then  $N$  is unitary and one may easily check that  $\Lambda_{j,k}N = \Lambda_{j+k,k}$ . As  $N$  does not alter the first coordinate of a triple  $\beta \in K^3$  then for  $k = \pm 1$

$$\sum_{\beta \in \Lambda_{j,k}} q^{|\beta|^2} z^{-3T(\beta_0/\lambda)} w^{T(\beta_0)}$$

is independent of  $j$ . Hence for  $k = \pm 1$

$$\begin{aligned} 3 \sum_{\beta \in \Lambda_{j,k}} q^{|\beta|^2} z^{-3T(\beta_0/\lambda)} w^{T(\beta_0)} &= \sum_{\beta \in (\mathcal{O}+k/\lambda)^3} q^{|\beta|^2} z^{-3T(\beta_0/\lambda)} w^{T(\beta_0)} \\ &= c_k(q, w, z^{-3})c(q)^2. \end{aligned} \quad (11)$$

From (9), (10) and (11) we obtain (5). The  $k = 0$  case of (5) is (6).  $\square$

**Corollary 1** *We have*

$$2a(q, z, w)^3 = b_1(q, w, z^{-3})b(q)^2 + b_{-1}(q, w, z^{-3})b(q)^2 + c_1(q, z, w)^3 + c_2(q, z, w)^3. \quad (12)$$

*Also*

$$a(q)^3 = b(q)^3 + c(q)^3. \quad (13)$$

**Proof** To obtain (12) subtract the sum the  $k = 1$  and  $k = -1$  cases of (5) from twice (6). To obtain (13), either substitute  $z = w = 1$  in (12), or make this substitution in either (6) or (5).  $\square$

Another particular case is obtained by setting  $w = 1$  to give

$$a(q, z, 1)^3 = b_1(q, 1, z^3)b(q)^2 + c_1(q, z, 1)^3$$

(using (4)) which is (1.25) in [3].

A variant of the argument of Theorem 1 gives the following result.

**Theorem 2** *For each  $k$ ,*

$$3c_k(q, z, w)c_k(q^2, z^2, w^2) = a(q, w, z^{-3})a(q^2) + \omega^k b_1(q, w, z^{-3})b(q^2) + \omega^{-k} b_{-1}(q, w, z^{-3})b(q^2) + c_1(q, w, z^{-3})c(q^2) + c_{-1}(q, w, z^{-3})c(q^2). \quad (14)$$

*In particular*

$$3a(q, z, w)a(q^2, z^2, w^2) = a(q, w, z^{-3})a(q^2) + b_1(q, w, z^{-3})b(q^2) + b_{-1}(q, w, z^{-3})b(q^2) + c_1(q, w, z^{-3})c(q^2) + c_{-1}(q, w, z^{-3})c(q^2). \quad (15)$$

**Proof** As the proof follows closely the proof of Theorem 1, we shall suppress most of the details.

Let  $V = \{(\alpha_0, \alpha_1, \alpha_2) \in K^3 : \alpha_1 = \alpha_2\}$ . The space  $V$  is stable under the action of the matrices  $M$  and  $N$ . The key is to rewrite the proof of Theorem 1 restricting the summations to triples in  $V$ . We start by noting that

$$c_k(q, z, w)c_k(q^2, z^2, w^2) = \sum_{\alpha \in (\mathcal{O} + 1/\lambda)^3 \cap V} \Phi(\alpha).$$

This gives

$$c_k(q, z, w)c_k(q^2, z^2, w^2) = \sum_{\beta \in (\Lambda_{-k, 0} \cap V) \cup (\Lambda_{-k, 1} \cap V) \cup (\Lambda_{-k, -1} \cap V)} q^{|\beta|^2} z^{-3T(\beta_0/\lambda)} w^{T(\beta_0)}.$$

We then get

$$\begin{aligned} & 3 \sum_{\beta \in \Lambda_{-k,0} \cap V} q^{|\beta|^2} z^{T(-3\beta_0/\lambda)} w^{T(\beta_0)} \\ &= a(q, w, z^{-3})a(q^2) + \omega^k b_1(q, w, z^{-3})b(q^2) + \omega^{-k} b_{-1}(q, w, z^{-3})b(q^2) \end{aligned}$$

and for  $j = \pm 1$

$$3 \sum_{\beta \in \Lambda_{j,k} \cap V} q^{|\beta|^2} z^{-3T(\beta_0/\lambda)} w^{T(\beta_0)} = c_k(q, w, z^{-3})c(q^2).$$

The theorem then follows.  $\square$

**Corollary 2** *We have*

$$\begin{aligned} 2a(q, z, w)a(q^2, z^2, w^2) &= b_1(q, w, z^{-3})b(q)^2 + b_{-1}(q, w, z^{-3})b(q)^2 \\ &\quad + c_1(q, z, w)^3 + c_2(q, z, w)^3. \end{aligned}$$

*Also*

$$a(q)a(q^2) = b(q)b(q^2) + c(q)c(q^2).$$

**Proof** This follows from Theorem 2 in exactly the same way that Corollary 1 follows from Theorem 1.  $\square$

Another special case is

$$a(q, z, 1)a(q^2, z^2, 1) = b_1(q, 1, z^3)b(q^2) + c_1(q, z, 1)c_1(q^2, z^2, 1)$$

which is (1.26) in [3].

## References

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